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Flat bidifferential ideals and semi-Hamiltonian PDEs

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Abstract
In this paper, we consider a class of semi-Hamiltonian systems characterized by the existence of a special conservation law. The density and the current of this conservation law satisfy a second-order system of PDEs which has a natural interpretation in the theory of flat bidifferential ideals. The class of systems we consider contains important well-known examples of semi-Hamiltonian systems. Other examples, such as genus 1 Whitham modulation equations for KdV, are related to this class by a reciprocal transformation.

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1. Introduction

Bidifferential ideals play an important role in the theory of finite-dimensional integrable systems, in particular in the bi-Hamiltonian theory of separation of variables [6, 20].

Some recent results [1, 19] suggest that they have also some applications in the theory of infinite-dimensional integrable systems, in particular in the case of integrable quasilinear PDEs.

In this paper, following [19], we want to deepen the study of these applications in the case of diagonal integrable systems of quasilinear PDEs, the so-called semi-Hamiltonian systems.

Definition 1 [26]. *A diagonal system of PDEs of hydrodynamic type*

$$u^i_t = v^i(u)u^i_x \qquad i = 1, \dots, n \tag{1}$$

is called semi-Hamiltonian if the coefficients $v^i(u)$ satisfy the system of equations

$$\partial_j\left(\frac{\partial_k v^i}{v^i - v^k}\right) = \partial_k\left(\frac{\partial_j v^i}{v^i - v^j}\right) \qquad \forall i \neq j \neq k \neq i, \tag{2}$$

where $\partial_i = \frac{\partial}{\partial u^i}$.

Equations (2) are the integrability conditions for the system

$$\frac{\partial_j w^i}{w^i - w^j} = \frac{\partial_j v^i}{v^i - v^j}, \tag{3}$$

which provides the characteristic velocities w^i of the symmetries of (1), and for the system

$$(v^i - v^j)\partial_i\partial_j H = \partial_i v^j \partial_j H - \partial_j v^i \partial_i H, \tag{4}$$

which provides the densities H of conservation laws of (1).

The knowledge of the symmetries of the system (1) allows one to find its general solution. Indeed, according to a general scheme of integration of semi-Hamiltonian systems proposed by Tsarev, the generalized hodograph method [26], any solution of a semi-Hamiltonian system is implicitly defined by a system of algebraic equations

$$w^i(u) = x + v^i(u)t \qquad i = 1, \ldots, n \tag{5}$$

where the functions $w^i(u)$ are the solutions of the system (3).

A classical result in the theory of first-order quasilinear PDEs [18] states that, if system (1) possesses a conservation law

$$\partial_t H + \partial_x K = 0$$

then the characteristic velocities v^i can be written in the form

$$v^i = -\frac{\partial_i K}{\partial_i H} \qquad i = 1, \ldots, n. \tag{6}$$

This result has some interesting consequences in the case of a semi-Hamiltonian systems. Due to integrability conditions (2) the space of solutions w^i of the system (3) is parametrized by n arbitrary functions of one variable.

Since the system (4) is invariant with respect to the substitution $v^i \to w^i$, for any solution $(w^1, \ldots, w^n)$ of the system (3) there exists a function K' such that

$$w^i = -\frac{\partial_i K'}{\partial_i H} \qquad i = 1, \ldots, n. \tag{7}$$

In other words the characteristic velocities of the symmetries can be obtained applying the linear operator

$$v^i_H(\cdot) := -\frac{1}{\partial_i H}\partial_i(\cdot) \qquad i = 1, \ldots, n, \tag{8}$$

to a suitable current K'.

Note that, in terms of the density H and of the currents K and K', the system of algebraic equations (5) reads

$$d(K' + xH - tK) = 0.$$

Substituting (7) in (3) and taking into account (4) we obtain the equations for the currents:

$$\partial_i\partial_j K' = \frac{\partial_j H}{\partial_i H}\frac{\partial_i v^j}{v^i - v^j}\partial_i K' - \frac{\partial_i H}{\partial_j H}\frac{\partial_j v^i}{v^i - v^j}\partial_j K'. \tag{9}$$

In general, the problem of finding the solutions of the system (9) could be very difficult. The aim of the present paper is to study a special class of semi-Hamiltonian systems characterized by the existence of a density of conservation law H such that equations (9) for the associated currents reduce to the form

$$(f^i - f^j)\partial_i\partial_j K' = \partial_i g^i \partial_j K' - \partial_j g^j \partial_i K', \tag{10}$$

where $f^i = f^i(u^i)$ and $g^i = g^i(u^i)$.

Surprisingly, also the density H is a solution of the system (10). Therefore the solutions of (10) play a double role.

- Fixed H, they are in one-to-one correspondence with the symmetries of a semi-Hamiltonian system (see formula (7)). In other words they define a semi-Hamiltonian hierarchy.
- They label these hierarchies: different choices of H correspond to different hierarchies.

The theory of flat bidifferential ideals arises naturally in this framework. First of all because any solution of the system (10) defines a flat bidifferential ideal. Second because it provides a recursive procedure to compute the solutions of (10).

The paper is organized as follows: in section 2 we recall some useful results about the theory of bidifferential ideals. In section 3, we apply these results to the theory of semi-Hamiltonian systems. Section 4 is devoted to a discussion of the Hamiltonian formalism. In particular we find a class of metrics satisfying a system of Egoroff–Darboux type. Remarkably, in general, these metrics are not related to any Frobenius manifold, since their rotation coefficients are not symmetric. Finally, in section 5, we put reciprocal transformations into the game.

2. Bidifferential ideals

A tensor field $L : TM \to TM$, of type (1, 1) on a manifold M, of dimension n, is torsionless if the following identity

$$[LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y] = 0$$

is verified for any pair of vector fields X and Y on M. According to the theory of graded derivations of Frölicher–Nijenhuis [11], a torsionless tensor field L of type (1, 1) defines a differential operator d_L, of degree 1 and type d, on the Grassmann algebra of differential forms on M, verifying the fundamental conditions:

$$d \cdot d_L + d_L \cdot d = 0 \qquad d_L^2 = 0.$$

On functions and 1-forms this derivation is defined by the following equations

$$d_L f(X) = \mathrm{d}f(LX)$$

$$d_L\alpha(X, Y) = \mathrm{Lie}_{LX}(\alpha(Y)) - \mathrm{Lie}_{LY}(\alpha(X)) - \alpha([X, Y]_L),$$

where

$$[X, Y]_L = [LX, Y] + [X, LY] - L[X, Y].$$

For instance, if $L = \mathrm{diag}(f^1(u^1), \ldots, f^n(u^n))$, the action of d_L on functions is given by the following formula:

$$d_L g := \sum_{i=1}^{n} f^i \frac{\partial g}{\partial u^i}\, \mathrm{d}u^i.$$

We can now define the concept of bidifferential ideal of forms.

Definition 2. *A bidifferential ideal $\Im$ is an ideal of differential forms on M which is closed with respect to the action of both d and d_L:*

$$d(\Im) \subset \Im \qquad d_L(\Im) \subset \Im.$$

For instance, if the ideal $\Im$ is generated by a single 1-form α, the condition of closure with respect to the action of d and d_L reads

$$\mathrm{d}\alpha = \lambda \wedge \alpha, \qquad d_L\alpha = \mu \wedge \alpha,$$

where λ and μ are suitable 1-forms.

In this paper, we need a special subclass of bidifferential ideals, called flat bidifferential ideals.

Definition 3. *A flat bidifferential ideal $\Im$, of rank 1, on a manifold M endowed with a torsionless tensor field $L : TM \to TM$, is the ideal of forms generated by the differential* $\mathrm{d}h$ *of a function $h : M \to \mathbb{R}$ obeying the condition*

$$dd_L h = \mathrm{d}h \wedge \mathrm{d}a \tag{11}$$

with respect to a function a which satisfies the cohomological condition

$$dd_L a = 0. \tag{12}$$

Remark 1. In the language of Dimakis and Müller–Hoissen, the pair $(d, d_L + \mathrm{d}a\wedge)$ defines a 'gauged bidifferential calculus'. Some applications of this calculus to the theory of integrable systems are discussed in [1].

From now on, if not stated otherwise, we assume that the eigenvalues of L are pairwise distinct. In this case, the general solution of equation (12) is given by the sum of n arbitrary functions of one variable:

$$a = \sum_{i=1}^{n} g^i(u^i)$$

and the cohomological equation (11) reads

$$(f^i - f^j)\partial_i\partial_j h = \partial_j h \partial_i g^i - \partial_i h \partial_j g^j \tag{13}$$

where $f^i(u^i)$ and $g^i = g^i(u^i)$ are arbitrary functions of one variable.

3. Flat bidifferential ideals and semi-Hamiltonian systems

In this section, we show that the linear operator (8) establishes a one-to-one correspondence between the space of solutions of the cohomological equation (11) (which is parametrized by n arbitrary functions of one variable) and the space of symmetries of a semi-Hamiltonian system. Indeed, it is easy to prove the following proposition.

Proposition 1. *Let $H(u)$ be a solution of the cohomological equation (11), then:*

(1) the systems

$$u^i_t = v^i_H(K_1)u^i_x, \qquad i = 1, \ldots, n, \tag{14}$$

$$u^i_\tau = v^i_H(K_2)u^i_x, \qquad i = 1, \ldots, n \tag{15}$$

commute for any pair (K_1, K_2) of solutions of (11);

(2) the system of quasilinear PDEs

$$u^i_t = \left[v^i_H(K)\right]u^i_x = \left[-\frac{\partial_i K}{\partial_i H}\right]u^i_x, \qquad i = 1, \ldots, n \tag{16}$$

is semi-Hamiltonian for any solution K of equation (11).

Proof.

(1) The commutativity condition for the systems (14) and (15) reads

$$\frac{\partial_j v^i_H(K_1)}{v^i_H(K_1) - v^j_H(K_1)} = \frac{\partial_j v^i_H(K_2)}{v^i_H(K_2) - v^j_H(K_2)}.$$

By straightforward computation we get

$$\frac{\partial_j v_H^i(K)}{v_H^i(K) - v_H^j(K)} = \frac{\partial_j H}{\partial_i H} \frac{\partial_i \partial_j H \partial_i K - \partial_i H \partial_i \partial_j K}{\partial_j K \partial_i H - \partial_i K \partial_j H} = \frac{\partial_i a}{f^j - f^i} \frac{\partial_j H}{\partial_i H}, \tag{17}$$

which does not depend on K.

(2) By definition of semi-Hamiltonian system we have to check that the characteristic velocities $v_H^i(K)$ satisfy the system (2). For $i \neq k \neq j \neq i$, we obtain the identity:

$$\partial_k \left(\frac{\partial_j v_H^i(K)}{v_H^i(K) - v_H^j(K)} \right) = \partial_k \left(\frac{\partial_i a}{f^j - f^i} \frac{\partial_j H}{\partial_i H} \right)$$
$$- \frac{\partial_i a}{(\partial_i H)^2} \left[\frac{\partial_j a \partial_i H \partial_k H}{(f^i - f^j)(f^j - f^k)} + \frac{\partial_k a \partial_i H \partial_j H}{(f^k - f^i)(f^j - f^k)} + \frac{\partial_i a \partial_j H \partial_k H}{(f^i - f^j)(f^k - f^i)} \right],$$

which is clearly symmetric w.r.t. the indices j and k. □

We have constructed a family of semi-Hamiltonian systems depending on functional parameters: the eigenvalues $f^i(u^i)$ of L, the functions a and H. If $\partial_i f_i \neq 0$, $(i = 1, \ldots, n)$, without loss of generality, we can assume $f^i(u^i) = u^i$; the two cases being simply related by the change of coordinates $u^i \to f^i(u^i)$.

Clearly, in order to make effective the construction one has to solve the cohomological equation (11). Even if its general solution, depending on n arbitrary functions of one variable, is known explicitly only in some special cases (see section 6 of [22] and references therein), the double differential complex defined by the pair (d, d_L) allows one to construct iteratively a countable set of solutions.

Lemma 1. *Let K_0 be a solution of (11). Then, the functions K_l defined recursively by*

$$\mathrm{d}K_{l+1} = d_L K_l - K_l\, \mathrm{d}a, \tag{18}$$

satisfy equation (11).

The proof is based on standard arguments in the theory of bidifferential ideals [20, 1]. We report it for the convenience of the reader.

Let us start with the first step of the recursive procedure

$$\mathrm{d}K_1 = d_L K_0 - K_0\, \mathrm{d}a. \tag{19}$$

First of all, let us verify that the 1-form appearing in the right-hand side of (19) is closed. Indeed, since K_0 is a solution of (11), applying to the right-hand side of (19) the differential d we obtain

$$d(d_L K_0 - K_0\, \mathrm{d}a) = \mathrm{d}K_0 \wedge \mathrm{d}a - \mathrm{d}K_0 \wedge \mathrm{d}a = 0.$$

So the function K_1 is (locally) well defined. Moreover

$$d d_L K_1 = d_L K_0 \wedge \mathrm{d}a = \mathrm{d}K_1 \wedge \mathrm{d}a.$$

We prove now the theorem by induction. Suppose that

$$\mathrm{d}K_l = d_L K_{l-1} - K_{l-1}\, \mathrm{d}a$$
$$d d_L K_l = \mathrm{d}K_l \wedge \mathrm{d}a.$$

Then the 1-form in the right-hand side of (18) is closed:

$$d(d_L K_l - K_l\, \mathrm{d}a) = \mathrm{d}K_l \wedge \mathrm{d}a - \mathrm{d}K_l \wedge \mathrm{d}a = 0$$

and satisfies equation (11):

$$d d_L K_{l+1} = d_L K_l \wedge \mathrm{d}a = \mathrm{d}K_{l+1} \wedge \mathrm{d}a.$$

Let us illustrate how to apply the previous procedure in the case $H = a$, $K_0 = -a$. Using the recursive relations (18) we get

$$
\begin{aligned}
u^i_{t_0} &= -\frac{\partial_i K_0}{\partial_i a} u^i_x = u^i_x \\
u^i_{t_1} &= -\frac{\partial_i K_1}{\partial_i a} u^i_x = [f^i - a]u^i_x = [f^i + K_0]u^i_x \\
u^i_{t_2} &= -\frac{\partial_i K_2}{\partial_i a} u^i_x = [(f^i)^2 + K_0 f^i + K_1]u^i_x \\
&\vdots \\
u^i_{t_n} &= -\frac{\partial_i K_n}{\partial_i a} u^i_x = [(f^i)^n + K_0(f^i)^{n-1} + K_1(f^i)^{n-2} + \cdots + K_{n-1}]u^i_x
\end{aligned}
$$

Following [19] we can write the above hierarchy in the coordinate-free form:

$$
\begin{aligned}
u_{t_0} &= u_x \\
u_{t_1} &= [L + K_0 E]^i u_x \\
u_{t_2} &= [L^2 + K_0 L + K_1 E]u_x \\
&\vdots \\
u_{t_n} &= [L^n + K_0 L^{n-1} + K_1 L^{n-2} + \cdots + K_{n-1}E]u_x
\end{aligned}
$$

where u is the column vector $(u^1, \ldots, u^n)^t$, E is the identity matrix and L is a torsionless tensor field of type $(1, 1)$. The above vector fields commute also in the non-diagonalizable case [19].

Example 1. $H = a$, $L = \mathrm{diag}(u^1, \ldots, u^n)$, $a = c\,\mathrm{Tr}(L)$

$$
\begin{aligned}
K_0 &= -a = -c\sum_j u^j \\
K_1 &= -\frac{1}{2}c\sum_j (u^j)^2 + \frac{1}{2}c^2\left(\sum_j u^j\right)^2 \\
K_2 &= -\frac{c}{3}\sum_j (u^j)^3 + \frac{c^2}{2}\sum_j (u^j)^2 \sum_j u^j - \frac{c^3}{6}\left(\sum_j u^j\right)^3
\end{aligned}
$$

and so on.

Example 2. (non-diagonalizable case). Let

$$
L = \begin{bmatrix} u^3 & \frac{u^2}{2} & 0 \\ 0 & u^3 & \frac{u^2}{2} \\ 0 & 0 & u^3 \end{bmatrix}.
$$

The function $a = u^1(u^2)^2$ satisfies the cohomological equation $dd_L a = 0$. Therefore the first non-trivial flow of the hierarchy starting from $K_0 = -a$ is

$$
\begin{bmatrix} u^1_t \\ u^2_t \\ u^3_t \end{bmatrix} = \begin{bmatrix} u^3 - a & \frac{u^2}{2} & 0 \\ 0 & u^3 - a & \frac{u^2}{2} \\ 0 & 0 & u^3 - a \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x . \\ u^3_x \end{bmatrix}
$$

The other non-trivial flows can be obtained solving the recursive relations (18) for the functions $K_1, K_2, \ldots$:

$$
\begin{aligned}
K_1 &= -u^1(u^2)^2u^3 - \tfrac{1}{8}(u^2)^4 + \tfrac{1}{2}(u^1)^2(u^2)^4 \\
K_2 &= -u^1(u^2)^2(u^3)^2 - \tfrac{1}{4}(u^2)^4u^3 + (u^1)^2(u^2)^2u^3 \\
&\quad - \tfrac{1}{6}(u_1)^3(u_2)^6 + \tfrac{1}{8}u_1(u_2)^6
\end{aligned}
$$

and so on.

Remark 2. Semi-Hamiltonian systems of the form

$$u^i_t = (f^i - a)u^i_x, \tag{20}$$

have been obtained in [22] as finite component reduction of an infinite hydrodynamic chain. The connection between bidifferential ideals and such systems has been investigated in [19]. The starting point of that paper was the observation that the conditions (4) and (2) for systems (20) coincide with the cohomological equations

$$dd_L H = \mathrm{d}a \wedge \mathrm{d}H$$

$$dd_L a = 0.$$

4. Some remarks about the Hamiltonian structure

The Hamiltonian formalism for systems of hydrodynamic type was introduced by Dubrovin and Novikov in [3, 4]. They considered first-order differential operators of the form

$$P^{ij} = g^{ij}(u)\partial_x - g^{is}\Gamma^j_{sk}(u)u^k_x \tag{21}$$

and the associated Poisson brackets

$$\{F, G\} := \int \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j}\, \mathrm{d}x \tag{22}$$

where $F = \int g(u)\, \mathrm{d}x$ and $G = \int g(u)\, \mathrm{d}x$ are local functionals.

Theorem 1 [3]**.** *If* $\det g^{ij} \neq 0$*, then formula (22) defines a Poisson bracket if and only if the tensor* g^{ij} *defines a flat pseudo-Riemannian metric and the coefficients* Γ^j_{sk} *are the Christoffel symbols of the associated Levi-Civita connection.*

Nonlocal extensions of the bracket (22), related to metrics of constant curvature, were considered by Ferapontov and Mokhov in [8]. Further generalizations were considered by Ferapontov in [7].

Let us focus our attention on semihamiltonan systems (1)

$$u^i_t = v^i(u)u^i_x, \qquad i = 1, \ldots n.$$

In [7] Ferapontov conjectured that any semi-Hamiltonian system is always Hamiltonian with respect to suitable, possibly nonlocal, Hamiltonian operators. Moreover he proposed the following construction to define such Hamiltonian operators:

1. Find the general solution of the system

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i}. \tag{23}$$

To this purpose is sufficient to find one solution g_{ii} of (23). Indeed, the general solution is $\frac{g_{ii}}{\varphi^i(u^i)}$, where φ^i are arbitrary functions of one argument. The flat solutions of (23) provide the local Hamiltonian structures of the system (1).

2. Write the non-vanishing components of the curvature tensor in terms of solutions w^i_α of the linear system (3):

$$R^{ij}_{ij} = \sum_\alpha \epsilon_\alpha w^i_\alpha w^j_\alpha \qquad \epsilon_\alpha = \pm 1. \tag{24}$$

(Ferapontov conjectured that it is always possible to find expansion (24).)

Then the system (1) is automatically Hamiltonian with respect to the Hamiltonian operator

$$P^{ij} = g^{ii}\delta^{ij}\partial_x - g^{ii}\Gamma^j_{ik}(u)u^k_x + \sum_\alpha \epsilon_\alpha w^i_\alpha u^i_x \partial_x^{-1} w^j_\alpha u^j_x. \tag{25}$$

Example 3. [7, 27, 21] Let us consider the system of chromatography equations in Riemann invariants

$$u^i_t = \left[u^i \prod u^k\right]^{-1} u^i_x \qquad i = 1, \dots n. \tag{26}$$

The general solution of (23), in this case, is

$$g_{ii} = \frac{\prod_{k\neq i}(u^k - u^i)^2}{\varphi^i(u^i)} \qquad i = 1, \dots, n,$$

where $\varphi^i(u^i)$ are n arbitrary functions of one variables. For $n \geqslant 3$ all these metrics are not flat [21]. They generate nonlocal Hamiltonian operators of the form [7]

$$P^{ij} = g^{ii}\delta^{ij}\partial_x - g^{is}\Gamma^j_{sk}(u)u^k_x - \sum_{\alpha=1}^{n} w^i_\alpha u^i_x \partial_x^{-1} w^j_\alpha u^j_x, \tag{27}$$

where

$$w^i_1 = \partial_i\left(\frac{\sqrt{\varphi^1}}{\prod_{l\neq 1}(u^l - u^1)^2}\right), \dots w^i_n = \partial_i\left(\frac{\sqrt{\varphi^n}}{\prod_{l\neq n}(u^l - u^n)^2}\right). \tag{28}$$

Note that system (26) can be written in the form

$$u^i_t = -\frac{\partial_i K}{\partial_i a} u^i_x, \qquad i = 1, \dots, n, \tag{29}$$

where

$$K = -\frac{1}{\prod_{k=1}^n u^k}$$

is a solution of the cohomological equation (11) with $L = \mathrm{diag}(u^1, \dots u^n)$, $a = -\mathrm{Tr}(L)$.

Example 4 [9]**.** Let us consider the semi-Hamiltonian system

$$u^i_t = \left[\sum_{i=1}^n u^i + 2u^i\right] u^i_x, \qquad i = 1, \dots n. \tag{30}$$

The general solution of (23) is

$$g_{ii} = \frac{\prod_{k\neq i}(u^k - u^i)}{\varphi^i(u^i)}, \qquad i = 1, \dots, n,$$

where $\varphi^i(u^i)$ are n arbitrary functions of one variables. The choice $\varphi^i(u^i) = (u^i)^\alpha$ $(\alpha = 0, \dots, n)$ provides $n+1$ flat metrics. For generic $\varphi^i(u^i)$ the metric g_{ii} is not flat and generates nonlocal Hamiltonian operator with infinite nonlocal tail. Note that system (30) can be written in the form (29) where

$$K = \frac{1}{4}\sum_j (u^j)^2 + \frac{1}{8}\left(\sum_j u^j\right)^2,$$

is a solution of the cohomological equation (11) with $L = \mathrm{diag}(u^1, \dots u^n)$, $a = -\frac{1}{2}\,\mathrm{Tr}(L)$.

Let us consider semi-Hamiltonian systems of the form (16). Taking into account equation (11), the system (23) reduces to

$$\frac{1}{2}\partial_i \ln g_{jj} = -\partial_j \ln \partial_i H - \frac{\partial_j a}{f^i - f^j}. \tag{31}$$

From now on, in this section, we assume $a = c\,\mathrm{Tr}(L) = c\sum_{j=1}^n f^j$. In this case, the general solution of (31) is

$$g_{ii} = \frac{(\partial_i H)^2}{\varphi^i(u^i)\left[\prod_{l\neq i}(f^i - f^l)\right]^{2c}}, \qquad i = 1, \dots n, \tag{32}$$

where $\varphi^i(u^i)$ are n arbitrary functions of one variable.

The rotation coefficients of the metrics (32) depend on the constant c, the eigenvalues $(f^1, \dots, f^n)$ and on the choice of the arbitrary functions $(\varphi_1, \dots, \varphi_n)$ but not on the function H. More precisely we have the following proposition.

Proposition 2. *Let H be a solution of the system:*

$$(f^i - f^j)\partial_i\partial_j H = c\partial_i f^i \partial_j H - c\partial_j f^j \partial_i H \tag{33}$$

Then, if $a = c\,\mathrm{Tr}(L)$, the rotation coefficients $\beta_{ij}(u) = \frac{\partial_i\sqrt{g_{jj}(u)}}{\sqrt{g_{ii}(u)}}$ of the metrics (32), does not depend on H. More precisely they are given by the following expression:

$$\beta_{ij} = \left[\frac{\prod_{l\neq i}(f^i - f^l)}{\prod_{l\neq j}(f^j - f^l)}\right]^c \frac{c\partial_j f^j}{f^j - f^i}\sqrt{\frac{\varphi_i}{\varphi_j}}. \tag{34}$$

Proof.

$$\beta_{ij}(u) = \sqrt{\frac{\varphi_i}{\varphi_j}}\left[\frac{\prod_{l\neq i}(f^i - f^l)}{\prod_{l\neq j}(f^j - f^l)}\right]^c \left[\frac{\partial_i\partial_j H}{\partial_i H} + c\frac{\partial_j H}{\partial_i H}\frac{\partial_j f_j}{f^j - f^i}\right].$$

Taking into account (33), we obtain formula (34). □

The problem of finding expansion (24) for the metrics (32) is, in general, very difficult. For $H = a$ this problem has been solved only for $c = \pm 1, 1/2$ (see section 9 of [22]).

The case $H \neq a$, which to our best knowledge, has not previously considered in the literature, can be reduced to the case $H = a = c\,\mathrm{Tr}(L)$. Indeed, we have the following proposition.

Proposition 3. *Let H be a solution of the system*

$$(f^i - f^j)\partial_i\partial_j H = c\partial_i f^i \partial_j H - c\partial_j f^j \partial_i H$$

and

$$u^i_{t_\alpha} = \tilde{w}^i_\alpha u^i_x = -\frac{\partial_i K_\alpha}{\partial_i H} u^i_x, \tag{35}$$

the corresponding semi-Hamiltonian hierarchy constructed with the solutions K_α of the system (33). Suppose that the hierarchy

$$u^i_{t_\alpha} = w^i_\alpha u^i_x = -\frac{\partial_i K_\alpha}{c\partial_i f^i} u^i_x \tag{36}$$

is Hamiltonian w.r.t. the Hamiltonian operator

$$P^{ij} = g^{ii}\delta^{ij}\partial_x - g^{ii}\Gamma^j_{ik}(u)u^k_x - \sum_\alpha w^i_\alpha u^i_x \partial_x^{-1} w^j_\alpha u^j_x \tag{37}$$

and the Hamiltonian densities h_α. Then the hierarchy (35) is Hamiltonian w.r.t. the Hamiltonian operator

$$\tilde{P}^{ij} = \tilde{g}^{ii}\delta^{ij}\partial_x - \tilde{g}^{ii}\tilde{\Gamma}^j_{ik}(u)u^k_x - \sum_\alpha \tilde{w}^i_\alpha u^i_x \partial_x^{-1} \tilde{w}^j_\alpha u^j_x, \tag{38}$$

where $\tilde{g}^{ii} = \left[c\frac{\partial_i f^i}{\partial_i H}\right]^2 g^{ii}$, the coefficients $\tilde{\Gamma}^j_{ik}$ are the Christoffel symbols of the associated Levi-Civita connection and $\tilde{w}^i_\alpha = \frac{c\partial_i f^i}{\partial_i H} w^i_\alpha$. Moreover the Hamiltonian densities $\tilde{h}_\alpha$ of the systems (35) can be obtained from the Hamiltonian densities h_α solving the compatible system

$$\partial_i \tilde{h}_\alpha = \frac{\partial_i H}{\partial_i a}\partial_i h_\alpha. \tag{39}$$

Proof. The non-vanishing components of the curvature tensor

$$R^{ij}_{ij} = g^{ii}\left(\partial_j \Gamma^j_{ii} - \partial_i \Gamma^j_{ij} - \Gamma^j_{pi}\Gamma^p_{ij} + \Gamma^j_{pj}\Gamma^p_{ii}\right) \tag{40}$$

can be written in the form

$$R^{ij}_{ij} = \frac{1}{\partial_i H \partial_j H} S^{ij}_{ij} \tag{41}$$

where the quantities S^{ij}_{ij} do not depend on H. Indeed in terms of the rotation coefficients (that do not depend on H), formula (40) reads

$$R^{ij}_{ij} = -\frac{1}{\sqrt{g_{ii}}}\frac{1}{\sqrt{g_{jj}}}\left(\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k\neq i,j} \beta_{ki}\beta_{kj}\right).$$

Using this fact it is easy to obtain expansion (24) for the non-vanishing components of the curvature tensor of the metric $g'_{ii} = c^2\left(\frac{\partial_i H}{\partial_i f^i}\right)^2 g_{ii}$:

$$R'^{ij}_{ij} = c^2 \frac{\partial_i f^i \partial_j f^j}{\partial_i H \partial_j H} R^{ij}_{ij}.$$

Observe that the coefficients $\tilde{w}^i_\alpha = c\frac{\partial_i f^i}{\partial_i H} w^i_\alpha = -\frac{\partial_i K_\alpha}{\partial_i H}$ are characteristic velocities of symmetries of (35). Therefore the bivector (38) satisfies all Ferapontov conditions. Indeed:

- the diagonal metric $\tilde{g}_{ii}$ is a solution of the system (23);
- the coefficients $\tilde{\Gamma}^i{}_{jk}$ are, by definition, the Christoffel symbols of the associated Levi-Civita connection;
- the nonlocal tail of (38) is constructed with the characteristic velocities $\tilde{w}^i_\alpha$ appearing in the expansion of the non-vanishing components of the curvature tensor.

We have to show now that the function $\tilde{h}_\alpha$ are Hamiltonian densities. First of all we observe that they are well defined. Indeed the compatibility of the system (39) reads

$$\partial_i\partial_j h_\alpha - c\frac{\partial_i h_\alpha \partial_j f^j}{f^i - f^j} + c\frac{\partial_j h_\alpha \partial_i f^i}{f^i - f^j} = 0, \tag{42}$$

which is nothing but the system (4) for the densities of conservation law of the semi-Hamiltonian hierarchy (36). Moreover it is easy to check that if the functions h_α are solutions of the system (42), then the functions $\tilde{h}_\alpha$ are solutions of the system (4) for the densities of conservation laws of the semi-Hamiltonian hierarchy (35). □

We conclude this section mentioning an important property of the metrics (32) in the case $L = \mathrm{diag}(u^1, \ldots, u^n)$ and $\varphi_i = 1 (i = 1, \ldots, n)$.

Proposition 4. *If $f^i(u^i) = u^i$ and $\varphi_i = 1 (i = 1, \ldots, n)$, the rotation coefficients (34) satisfy the system*

$$\partial_k\beta_{ij} = \beta_{ik}\beta_{kj} \qquad i \neq j \neq k \tag{43}$$

$$\sum_k \partial_k\beta_{ij} = 0 \qquad i \neq j \tag{44}$$

$$\sum_k u^k\partial_k\beta_{ij} = -\beta_{ij} \qquad i \neq j. \tag{45}$$

Proof. Equations (43) are automatically satisfied because they are equivalent to the conditions (2).

Moreover, by straightforward computation we obtain

$$\begin{aligned}
\sum_k \partial_k\beta_{ij} &= \partial_j\beta_{ij} + \partial_i\beta_{ij} + \sum_{k\neq i,j}\beta_{ik}\beta_{kj} \\
&= \left[\frac{\prod_{l\neq i}(u^i - u^l)}{\prod_{l\neq j}(u^j - u^l)}\right]^{c-1} \frac{c^2}{u^j - u^i} \\
&\quad \times \left\{-\frac{\prod_{l\neq i,j}(u^i - u^l)}{\prod_{l\neq j}(u^j - u^l)} - \frac{\prod_{l\neq i}(u^i - u^l)\sum_{k\neq j}\prod_{l\neq j,k}(u^j - u^l)}{\prod_{l\neq j}(u^j - u^l)^2}\right. \\
&\quad \left. + \frac{\sum_{k\neq i}\prod_{l\neq i,k}(u^i - u^l)}{\prod_{l\neq j}(u^j - u^l)} + \frac{\prod_{l\neq i}(u^i - u^l)\prod_{l\neq i,j}(u^j - u^l)}{\prod_{l\neq j}(u^j - u^l)^2}\right\} \\
&\quad + \left[\frac{\prod_{l\neq i}(u^i - u^l)}{\prod_{l\neq j}(u^j - u^l)}\right]^{c} \frac{c^2}{u^j - u^i}\sum_{i,j}\left[\frac{1}{u^k - u^i} + \frac{1}{u^j - u^k}\right] \\
&= \left[\frac{\prod_{l\neq i}(u^i - u^l)}{\prod_{l\neq j}(u^j - u^l)}\right]^{c} \frac{c^2}{u^j - u^i} \\
&\quad \times \left\{-\sum_{k\neq i,j}\frac{1}{u^j - u^k} + \sum_{k\neq i,j}\frac{1}{u^i - u^k} + \sum_{k\neq i,j}\left(\frac{1}{u^k - u^i} + \frac{1}{u^j - u^k}\right)\right\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\sum_k u^k\partial_k\beta_{ij} &= \sum_{k\neq i,j} u^k\beta_{ik}\beta_{kj} + u^j\partial_j\beta_{ij} + u^i\partial_i\beta_{ij}\\
&= \left[\frac{\prod_{l\neq i}(u^i-u^l)}{\prod_{l\neq j}(u^j-u^l)}\right]^c \left\{\frac{c^2u^k}{u^j-u^i}\sum_{k\neq i,j}\left[\frac{1}{u^k-u^i}+\frac{1}{u^j-u^k}\right]-\frac{c}{u^j-u^i}\right.\\
&\quad\left.+\frac{c^2}{u^j-u^i}\left[\frac{u^j}{u^j-u^i}-\sum_{k\neq j}\frac{u^j}{u^j-u^k}+\sum_{k\neq i}\frac{u^i}{u^i-u^k}+\frac{u^i}{u^j-u^i}\right]\right\}\\
&= -\beta_{ij}+\left[\frac{\prod_{l\neq i}(u^i-u^l)}{\prod_{l\neq j}(u^j-u^l)}\right]^c\frac{c^2}{u^j-u^i}\\
&\quad\times\left\{\sum_{k\neq i,j}\left[\frac{u^k}{u^k-u^i}+\frac{u^k}{u^j-u^k}\right]-\sum_{k\neq i,j}\frac{u^j}{u^j-u^k}+\sum_{k\neq i,j}\frac{u^i}{u^i-u^k}\right\}\\
&= -\beta_{ij}+\left[\frac{\prod_{l\neq i}(u^i-u^l)}{\prod_{l\neq j}(u^j-u^l)}\right]^c\frac{c^2}{u^j-u^i}\left\{\sum_{k\neq i,j}\left[\frac{u^k-u^j}{u^j-u^k}+\frac{u^k-u^i}{u^k-u^i}\right]\right\}\\
&= -\beta_{ij}.
\end{aligned}$$

□

Remark 3. In general, the rotation coefficients (34) are not symmetric. In the case of symmetric rotation coefficients, equations (43)–(45) arise naturally in the framework of Frobenius manifolds [2].

As is well known, the theory of Frobenius manifolds is related to the theory of isomonodromic deformations. Indeed, equations (43)–(45) are equivalent to the system:

$$\partial_k V(u) = [V(u), [E_k, \Gamma]], \qquad k = 1, \dots, n, \tag{46}$$

where

$$\begin{aligned}
(E_k)_{ij} &= \delta_{ik}\delta_{kj}\\
U &:= \mathrm{diag}(u^1, \dots, u^n)\\
\Gamma(u) &:= (\beta_{ij})\\
V(u) &:= (u^i - u^j)\beta_{ij},
\end{aligned}$$

that governs the monodromy preserving deformations of the operator

$$\frac{d}{dz} - \left(U + \frac{V}{z}\right).$$

The proof of this fact (see [2]) does not rely on the hypothesis of symmetry of the rotation coefficients.

5. A remark on Whitham equations

A well-known example of semi-Hamiltonian system is the system of quasilinear PDEs that describes the slow modulations of g-gap solutions of the KdV hierarchy: the Whitham equations [28]. In this case, the characteristic velocities can be written in terms of hyperelliptic integrals of genus g. In $g = 1$ case these equations read

$$u^1_t = \left[-\frac{u^1+u^2+u^3}{3} + \frac{2(u^2-u^1)K(s)}{3(K(s)-E(s))}\right]u^1_x$$
$$u^2_t = \left[-\frac{u^1+u^2+u^3}{3} + \frac{2(u^2-u^1)K(s)}{3(E(s)-(1-s^2)K(s))}\right]u^2_x$$
$$u^3_t = \left[-\frac{u^1+u^2+u^3}{3} - \frac{2(u^3-u^1)(1-s^2)E(s)}{3E(s)}\right]u^3_x$$

where $s = \frac{u^2-u^1}{u^3-u^1}$, $K(s)$ and $E(s)$ are complete elliptic integrals of the first and second kind.

For Whitham equations, the hodograph method is effective [5, 12–16, 23–25]. Indeed it is possible to construct explicitly the symmetries appearing in equations (5).

Theorem 2 [13, 14, 16, 23, 24]**.** *There exist functions $q_1(u), q_2(u), q_3(u)$ such that the characteristic velocities w^i of the symmetries of the Whitham equations have the form*

$$w^i := [1 + q_i\partial_i]\,K, \qquad i = 1, \ldots, 3, \tag{47}$$

where the function K is a solution of the following system of Euler–Poisson–Darboux type:

$$2(u^i - u^j)\partial_i\partial_j K = \partial_i K - \partial_j K \qquad i \neq j, \quad i, j = 1, 2, 3. \tag{48}$$

which can be explicitly solved.

The functions $q_i(u)$ can be written in terms of the complete elliptic integral $K(s)$ and $E(s)$. Moreover from the conservation of waves it follows that [13, 14, 17, 25]:

$$q_i(u) = -\frac{H}{\partial_i H} \qquad i = 1, 2, 3,$$

where

$$H = \oint \frac{\mathrm{d}\xi}{\sqrt{(u^1-\xi)(u^2-\xi)(u^3-\xi)}} \tag{49}$$

is the wavelength (the integration is taken over the cycle around the gap $u^2 \leqslant \xi \leqslant u^3$). Therefore, the Whitham equations can be written in the form:

$$u^i_t = v^i_H(K) = \left[K - \frac{H}{\partial_i H}\partial_i K\right]u^i_x, \qquad i = 1, 2, 3.$$

Note that the wavelength H satisfies the Euler–Darboux–Poisson system (48), which is a particular case of the cohomological equation (11) corresponding to the choice $L = \mathrm{diag}(u^1, u^2, u^3)$, $a = -\frac{1}{2}\,\mathrm{Tr}(L)$. This remark suggests to consider systems of the form

$$u^i_t = v^i_H(K)u^i_x = \left[K - \frac{H}{\partial_i H}\partial_i K\right]u^i_x, \qquad i = 1, \ldots, n, \tag{50}$$

where the functions H and K are solutions of the cohomological equations (13). It is easy to prove that such systems are semi-Hamiltonian and that, fixed H, the systems

$$u^i_t = v^i_H(K_1)u^i_x, \qquad i = 1, \ldots, n$$
$$u^i_\tau = v^i_H(K_2)u^i_x, \qquad i = 1, \ldots, n,$$

commute for any pair (K_1, K_2) of solutions of (13).

This fact can be proved by straightforward calculation or simply observing that systems of the form (47) can be obtained from the systems studied in this paper by means of a reciprocal transformation.

Proposition 5. *Systems*

$$u^i_{\tilde{t}} = \left[K - \frac{H}{\partial_i H}\partial_i K\right] u^i_{\tilde{x}}, \qquad i = 1, \ldots, n,$$

are related to the systems

$$u^i_t = \left[-\frac{\partial_i K}{\partial_i H}\right] u^i_x$$

by the reciprocal transformation

$$\mathrm{d}\tilde{x} = H\,\mathrm{d}x - K\,\mathrm{d}t$$
$$\mathrm{d}\tilde{t} = \mathrm{d}t.$$

The proof is a trivial computation.

6. Conclusions

In this paper, we studied some applications of the theory of flat bidifferential ideals to semi-Hamiltonian systems of quasilinear PDEs.

The starting point of the present paper was the observation that for any semi-Hamiltonian system there exists a linear differential operator that, acting on a suitable domain, provides all the symmetries of the system.

We showed that for a special class of semi-Hamiltonian systems this operator and its domain are completely characterized by the solutions of a cohomological equation.

Moreover the theory of flat bidifferential ideals naturally provides a recursive procedure to compute the solutions of this equation.

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